V

MULTISTAGE GAMES

V.1 BEHAVIORAL STRATEGIES

Let us consider a game with many moves. It may be a game as simple, say, as tic-tac-toe. This game is simple enough so that even a child can learn to master it. Yet suppose we wish to count the number of strategies for the first player. We note (disregarding all symmetries) that he has nine choices for the first move. Then, for any of the eight possible replies, he will have seven choices on his second move. If we consider only the first player's first two moves, we find that he has $9 \cdot 7^8 = 51{,}883{,}209$ pure strategies. Any attempt to enumerate them, naturally, is out of the question. Even if we consider symmetries, we find that the number of pure strategies for this game is astronomical. And yet the game is quite trivial (in the practical sense of the word, as opposed to chess, which is theoretically trivial but practically very complex).

It is clear, then, that pure strategies leave something to be desired. Let us remember the definition of a pure strategy: it is a function, defined on the collection of a particular player's information sets, assigning to each information set a number between 1 and k (where k is the number of choices at the given information set). Thus if a player has N information sets, and k choices at each one, the total number of pure strategies is k^N, which can be very large.

Going back to the example of tic-tac-toe we find that no one plays the game by actually considering all possible pure strategies (i.e., all possible sequences of moves from the first to the last). Rather it is played by considering, at each move, all the possible choices *for that move only*, and deciding (from experience or otherwise) which is best.

This, then, is the essence of simplification: take the moves one at a time. It reduces one choice among $k_1 k_2 \cdots k_N$ possible strategies to N choices among the k_i possible moves at each information set. It leads to the following definition:

V.1.1 Definition A *behavioral strategy* is a collection of N probability distributions, one each over the set of possible choices at each information set.

V.1.2 Example A player is given a card from a deck of 52 cards; after seeing his card, he has a choice of either *passing* or *betting* a fixed amount. The total number of pure strategies is 2^{52}; thus, the set of mixed strategies will have dimension $2^{52} - 1$. On the other hand, a behavioral strategy simply gives the probability of betting (a number between 0 and 1) with each hand. Thus the dimension of the set of behavioral strategies is 52.

In general, then, the set of behavioral strategies is of much smaller dimension than the set of mixed strategies. On the other hand, it should be pointed out that under certain conditions, not all mixed strategies can be attained by using behavioral strategies, as can be seen by the following example:

V.1.3 Example In the game with tree in Figure V.1.1, we can label player II's pure strategies as LL, LR, RL, RR. A mixed strategy is a vector (x_1, x_2, x_3, x_4) satisfying the usual constraints; a behavioral strategy is a pair of numbers (y_1, y_2) satisfying only $0 \le y_i \le 1$. To the behavioral strategy (y_1, y_2), there corresponds the mixed strategy

$$(y_1, y_2, y_1(1 - y_2), (1 - y_1)y_2, (1 - y_1)(1 - y_2)). \qquad (5.1.1)$$

Now, the optimal strategy for player II is (see Example II.5.9) the vector $(0, \frac{5}{7}, \frac{2}{y}, 0)$. Clearly, this is not of the form (5.1.1). Hence the game does not admit a solution in terms of behavioral strategies.

In a sense, the difficulty with Example V.1.3 is that player II does not know, at his second move, what he did at his first move. Such a game is said to have *imperfect recall*. This difficulty complicates matters: it is not certain whether all mixed strategies are permitted. In bridge, the two partners are treated as a single player who "forgets" some of his previous choices. As secret arrangements are forbidden, one partner cannot know whether the bid made by his partner was honest or a "psychic" bid, although he may know the probability of such a "psychic" bid.

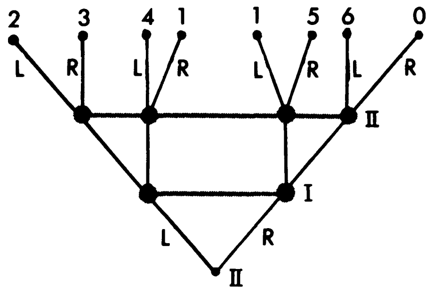

Figure V.1.1

V.2 GAMES OF EXHAUSTION

We shall not enter into these difficulties here. The type of game which we propose to study is very nearly a game of perfect information: after a given number of moves, both players are (simultaneously) given perfect information (which will not be forgotten) so that in a sense the game can be recommenced from the given position. The general pattern will be: first, player I moves; then, player II moves (in ignorance of player I's previous move); next, a random move is made, after which both players are given perfect information. (This pattern may vary, but only slightly.) Each of these cycles will be called a *stage* of the game.

It is advisable to solve these games by working backward. The general idea is that each stage can be treated as a separate game. When the strategies for this stage are chosen, the payoff will be either a true payoff (in case the multistage game terminates) or an obligation to play a subsequent stage of the game.

Since we generally deal with expected values, it follows that we can replace the obligation to play a game by the value of that particular game.

V.2.1 Example Consider the game with matrix

$$\begin{pmatrix} a_{11} & \Gamma_1 \\ \Gamma_2 & a_{12} \end{pmatrix}, \tag{5.2.1}$$

where the entries Γ_1 and Γ_2 represent the obligation to play two other games with the respective matrices:

$$\Gamma_1 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \qquad \Gamma_2 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}. \tag{5.2.2}$$

If the values of Γ_1 and Γ_2 are v_1 and v_2, respectively, it follows that, in expectation, the prospect of having to play these games is equivalent to